METRIC SPACES: FINAL EXAM 2016

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Evaluation: $\min\Big(100\%, \max(5\,\mathrm{prb} \times 20\% \cdot \left[\begin{smallmatrix}1.00\\ 1.15^{\mathrm{top}}\end{smallmatrix}\right], \sum_{i=1}^{6} \mathrm{h/w} \times 5\% + 5\,\mathrm{prb} \times 14\% \cdot \left[\begin{smallmatrix}1.00\\ 1.15^{\mathrm{top}}\end{smallmatrix}\right])\Big)$.

Problem 1. Let $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$ be a non-empty metric space, r and s be two positive radii, and $\mathsf{B}_r^{\mathrm{d}_{\mathcal{X}}}(x) = \mathsf{B}_s^{\mathrm{d}_{\mathcal{X}}}(y)$ for some $x, y \in \mathcal{X}$.

- Is it true that $r = s$?
- Is it true that $x = y$?

Problem 2. Let $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$ be a metric space and $\varnothing \neq A \subseteq \mathcal{X}$ its subset. Prove that the interior $\mathrm{Int}\,(A) = \{a \in A \mid \exists\, \varepsilon(a) > 0, \mathsf{B}_\varepsilon^{\mathrm{d}_{\mathcal{X}}}(a) \subseteq A\}$ is open in $\mathcal{X}$.

Problem 3. Let A and B be connected subsets of a metric space and $A \cap \overline{B} \neq \varnothing$. Prove that the union $A \cup B$ is connected.

Problem 4. Let A and B be compact subsets of a Hausdorff space $\mathcal{X}$. Prove that their intersection $A \cap B$ is compact.

Problem 5. Let $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$ be a non-empty complete metric space. Suppose that $f, g \colon \mathcal{X} \to \mathcal{X}$ are two Banach's contractions of $\mathcal{X}$. Prove that there always exists a unique point $x_0 \in \mathcal{X}$ such that $f\big(g(x_0)\big) = x_0$.

Date: April 1, 2016.

Do not postpone your success until 22 June. GOOD LUCK!